Linton Sets

Ternions

It is a well known fact that ternions (i.e. three dimensional numbers $\{x, y, z\}$ do not exist.

But not everything which is 'well known' is necessarily true.

What is true is that numbers of the form $x + iy + jz$ do not 'work' algebraically and are of no use in describing the behaviour of vectors in three dimensions.

On the other hand, in mathematics you can define anything you like and there is no objection to treating $\{x, y, z\}$ as a 'numerical' entity provided that you define operations like addition and multiplication in a consistent way.

For my purposes I only need to define addition (and subtraction) and squaring (and square rooting).

Addition and subtraction are achieved by simply adding or subtracting the X, Y and Z components. It is easy to see that this process in independent of the coordinate system chosen because it is tantamount to putting the two vectors end to end.

In two dimensions (i.e. the complex plane), multiplication is achieved by multiplying the moduli and adding the arguments. In three dimensions the situation is not clear because the two 'arguments' (azimuth and altitude) are not equivalent and are coordinate specific. On the other hand, we can plausibly define the square of a ternion, by analogy with that of a complex number, by squaring the modulus and doubling the argument where the argument is defined as the angle between the vector and the X axis (keeping the new vector in the same plane as the old one and the X axis). (This angle is sometimes called the polar latitude.)

The square root of a ternion is therefore seen to have two values: in which the new modulus is the square root of the original modulus and the argument is halved – the other root being the antipodean point.

There is one small problem with this definition. The square root of {-1, 0, 0} is undefined – or rather, there are an infinite number of possible roots

Julia Sets

The famous Mandelbrot Set and its associated Julia Sets are generated using the iterative formula

$$z' = z^2 + C \tag{1}$$

For any given value of C, there is a set of values of z which are stable under repeated iteration. Some of these sets are filled and some are fragmentary. Either way, the *border* of this set is the *Julia Set* for that value of C. These are the points which are only just stable. If you start at one of these points, z will hop round the set but if, for example, due to rounding errors, z strays slightly outside the set, it will diverge off to infinity. Likewise if is strays into a filled region it will become trapped inside. The Julia Set is unstable: it is a repeller.

A simple way of generating the Julia Set for any given point C is to pick a point at random and then apply the reverse iteration

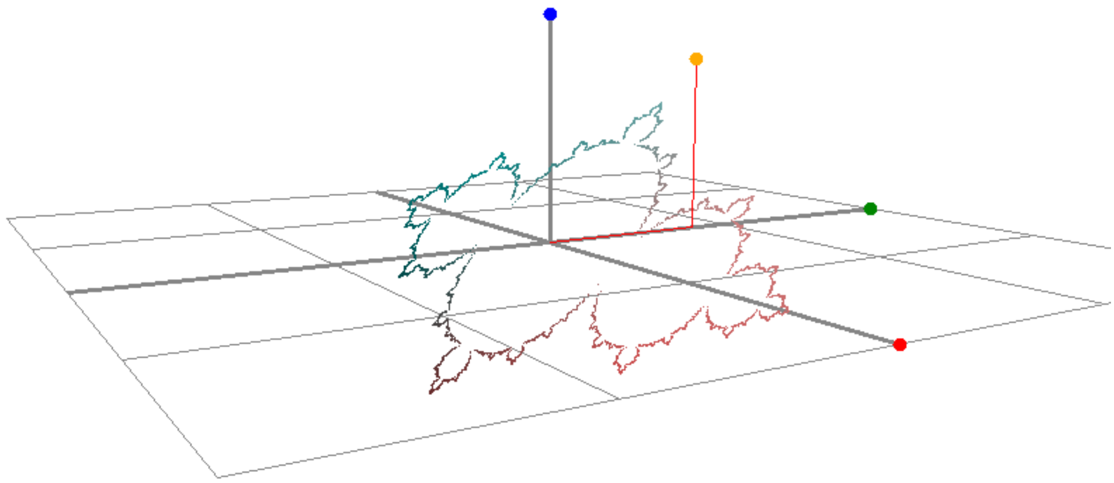
$$z' = \sqrt{z - C} \tag{2}$$

The Julia Set is now an attractor and after 100 or so iterations, you can be pretty sure you are very close to the actual set. Now, in order to generate the complete set, since the square root has two solutions, you should, in theory pursue both options. In practice it is sufficient to choose one of them at random and simply iterate thousands of times. There will be some points missing (e.g. the point which is reached after choosing the positive root 100 times) but the general shape soon becomes clear. It is sometimes a good idea to bias the choice towards the positive or negative root in order to ensure that these points do get visited occasionally.

Linton Sets

In order to generate a three dimensional Julia Set (which, with characteristic modesty, I shall call a Linton Set) we simply start with a random point P_0 and apply equation (2) using the square root function defined as above, plotting the resulting points in 3D. If in the unlikely event that z happens to jump to a point *exactly* on the negative X axis then the iterations are stopped and restarted with the original initial point.

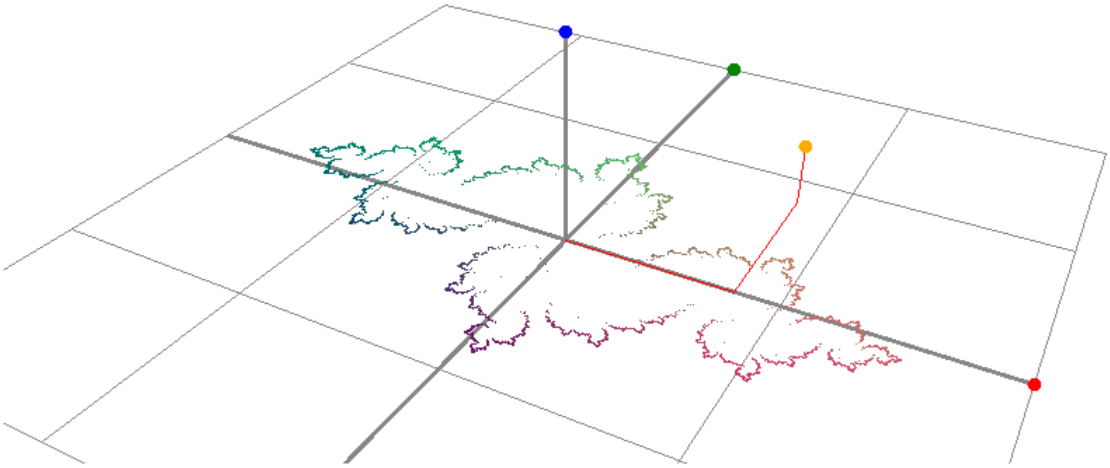
There are several special cases to discuss. First let us examine the Linton set for $C = \{-0.8, 0, 0\}$. In the illustration below the X axis is indicated by a red blob, Y, green, Z, blue and the initial point (which is equal to $\{0, 0.8, 0.8\}$ in this case) is indicated by an orange blob.



It is immediately obvious that the Linton set for this value of C is simply the standard Julia set for $C = (-0.8, 0)$ but tilted to lie in a plane which includes the initial starting point. It is not too difficult to see why. With the Y and Z coordinates of C both being zero, it is possible to rotate our coordinate system about the X axis such that the initial point lies in the XY' plane. Now with a Z' coordinate of zero, the point is constrained to move in the plane which is defined by the initial point P_0 and the X axis. (I shall call this plane the XP_0 plane)

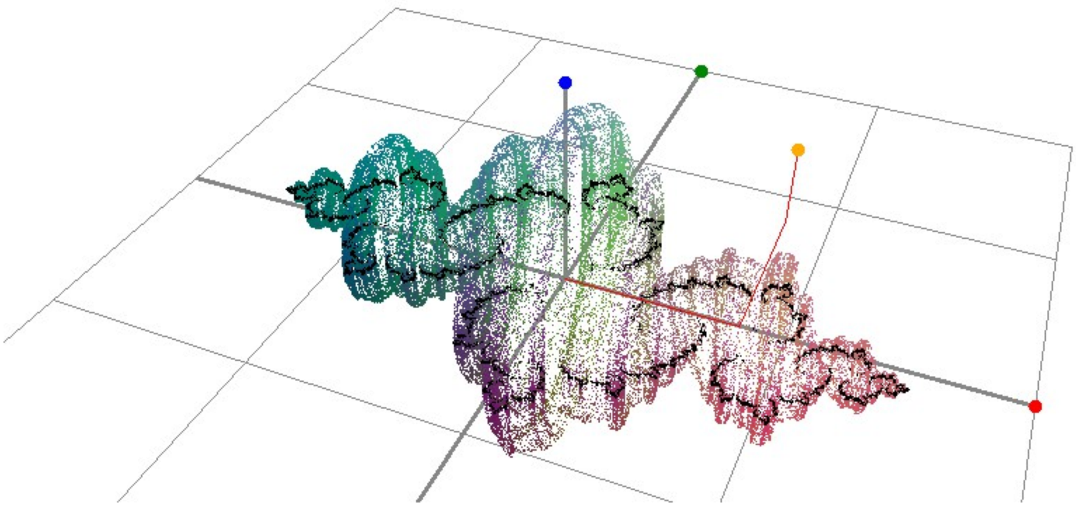
In fact, since we are at liberty to choose the Y and Z axes in any way we like, we can always consider the point C to lie in the XY plane – in other words, without loss of generality, we need only ever consider cases where $C.z = 0$.

Consider, for example the Linton set for $C = \{–0.8, 0.2, 0\}$. If P_0 lies in the XY plane, then subsequent iterations will always lie in that plane and what we get is the standard Julia set for the point $C = (–0.8, 0.2)$. In fact, even if $P_0.z$ is not equal to zero, the attractor still seems to work because the Z component seems to decay to zero and the resulting fractal still lies flat on the XY plane.



It is important to understand why the Z component decays. According to the definition of the square root of a ternion, the polar latitude is halved. Now when the polar latitude of *z* is less than 90° then the magnitude of the Z component will always decrease. In fact, even when the polar latitude is greater than 90° but less the 120°, the Z component will still decrease. Only when the polar latitude is greater than 120° will the Z component increase. Now the polar latitude can only be greater than 120° if the negative root was chosen after the previous iteration. It follows that if we choose the positive and negative roots equally often, the Z component of *z* will decrease more often than it will increase.

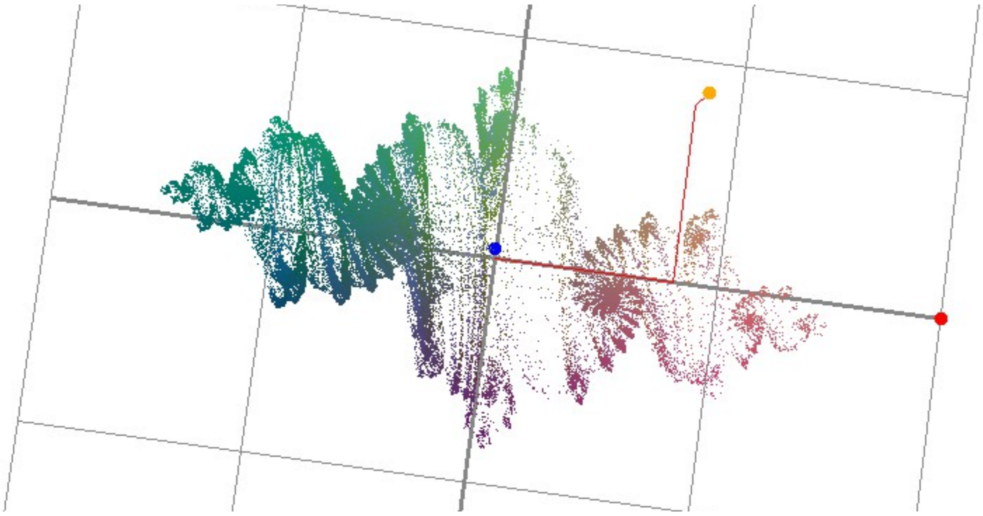
In order to see the full Linton set, it is necessary to bias the choice of root towards the negative root. This is what we get:



The full Linton set resembles the surface of a piece of Danish pastry which has been twisted.

Where the shape intersects the XY plane, we find the standard Julia set (emphasised here in black) but the Linton set is not simply the Julia set rotated about the X axis. (One reason for this is that, when the polar latitude of a point is halved in the XP_0 plane, the polar angle in the XY plane – or, indeed, any other plane – is not exactly halved. It follows that the if the Linton set is sliced along any other plane which includes the X axis, the shape will not be exactly the same.)

Looked at from a vertical viewpoint, multiple three dimensional twists and spirals can be discerned which have a counterpart in the familiar coloured versions of the Julia set.



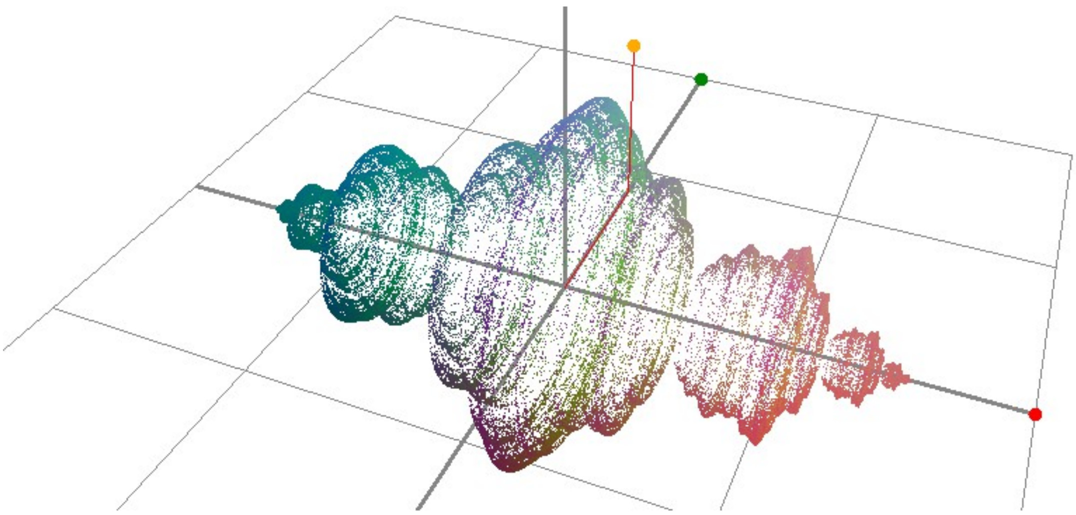
Like all strange attractors, the Linton set is independent of the starting point used.

On the other hand, we considered earlier the Linton set for the point {–0.8, 0, 0} and found that it did depend on the initial starting point. Of course this is a rather special case but it is worth considering further why this point does not generate a 3D Linton set.

In fact, it should. I mentioned earlier that the point {–1, 0, 0} does not have a square root – or rather, it has an infinite number of square roots. If at any time *z* should jump to a point exactly on the negative X axis, the next iteration could go anywhere on a circle in the YZ plane. The reason why this never happens is that *z.z* can only reach zero after an infinite number of successive positive roots – which is never going to happen.

We can, however, simulate the effect by adding a tiny bit of uncertainty into the calculations. Normally any inaccuracies are smoothed out by the continual halving and square rooting – but, if the point *z* happens to stray close to the negative X axis, small changes in the Y or Z coordinate can make very large changes to the longitude angle and hence to the subsequent location of *z*.

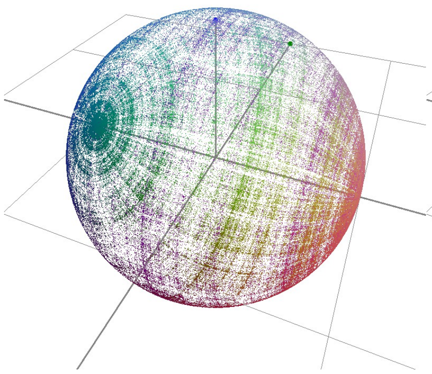
This is what we get when we add a bit of fuzziness even when the choice of root is 50:50..



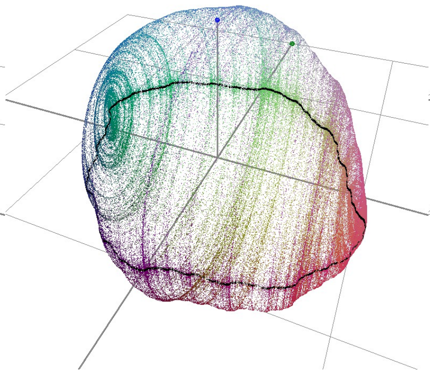
This time, however, the set is, of course, simply the surface of revolution of the Julia set.

What about the Linton set for the point *C* = {0, 0, 0}? The Julia set for *C* = (0, 0) is the unit circle. It will be no surprise, therefore, to learn that the Linton set for {0, 0, 0} is the unit sphere.

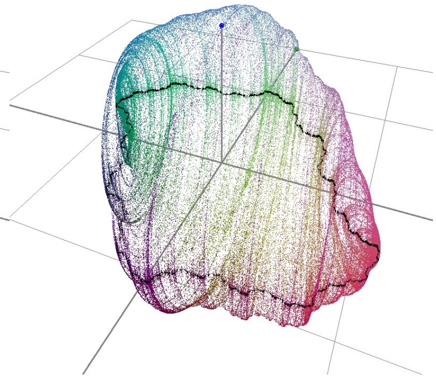
The following sequence of images shows how the fractal changes as *C* moves from the origin along the Y axis in steps of 0.2. (To achieve these images I have used both fuzzy arithmetic and a slight bias towards one or other root root.)



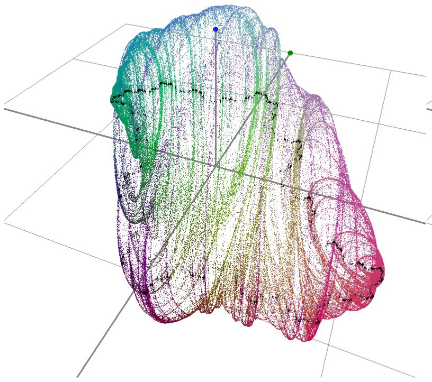
{0, 0, 0}



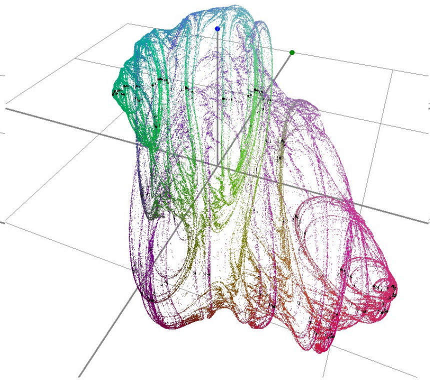
{0, 0.2, 0}



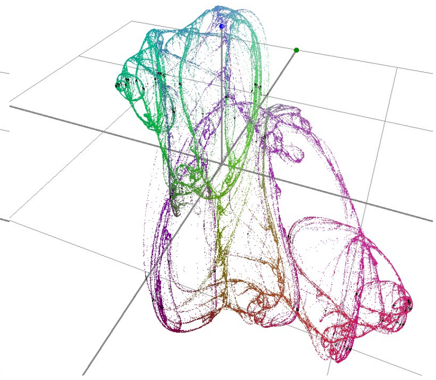
{0, 0.4, 0}



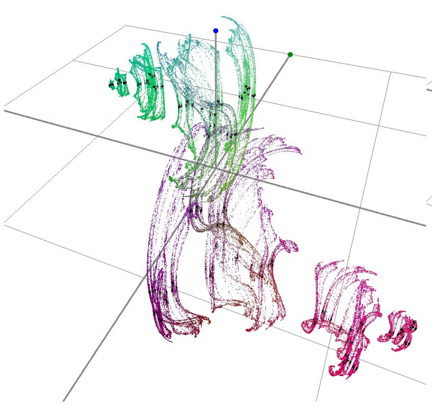
{0, 0.6, 0}



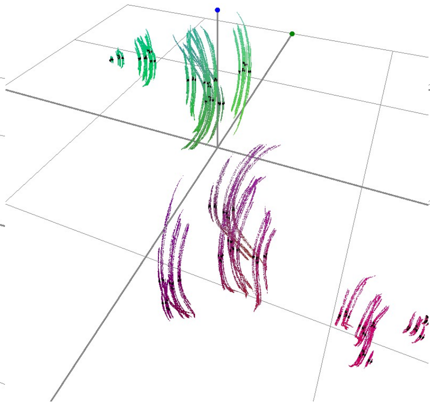
{0, 0.8, 0}



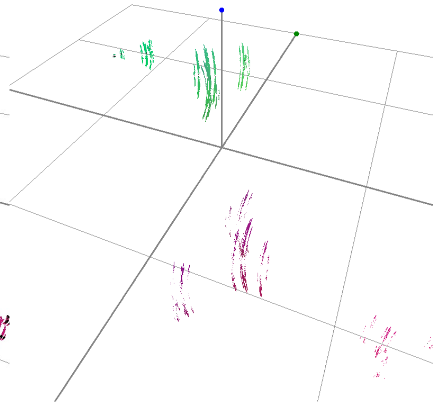
{0, 1, 0}



{0, 1.2, 0}



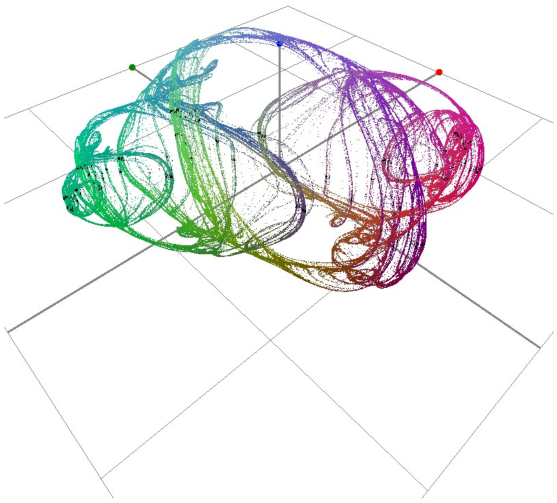
{0, 1.4, 0}



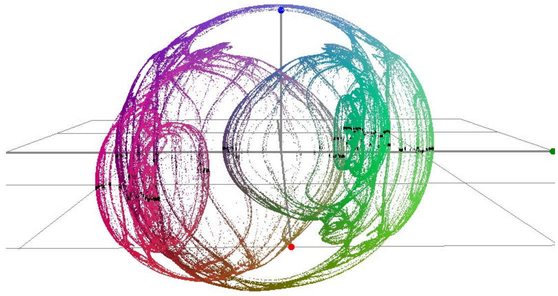
{0, 1.6, 0}

It is quite difficult to make out what is going on here. The three dimensional shapes are so complex that, even when the object is rotated in real time, it is difficult to see if and how the various parts are connected. At first the sphere is distorted and the outline of the intersection of the set and the XY plane is, as always, the standard Julia set. When $C.y =$ ~0.64, the standard Julia set breaks up into fragments and the Linton set splits into thin filaments, no longer enclosing a finite volume.

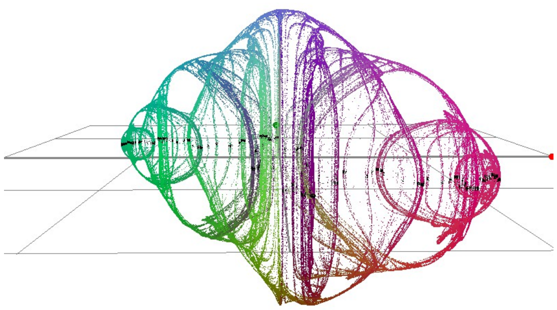
The following images show the Linton set for {0, 1, 0} from various other angles.



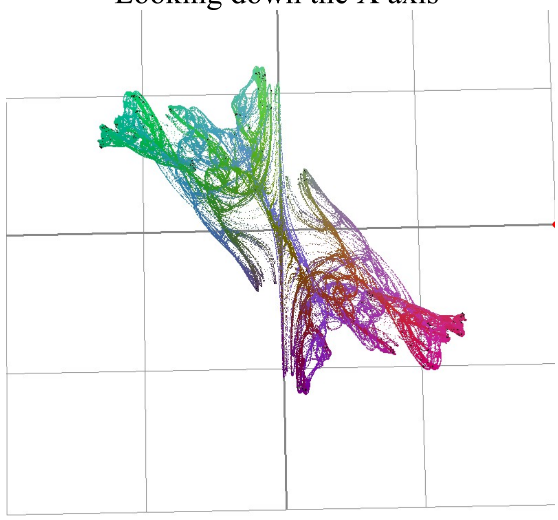
Oblique view



Looking down the X axis

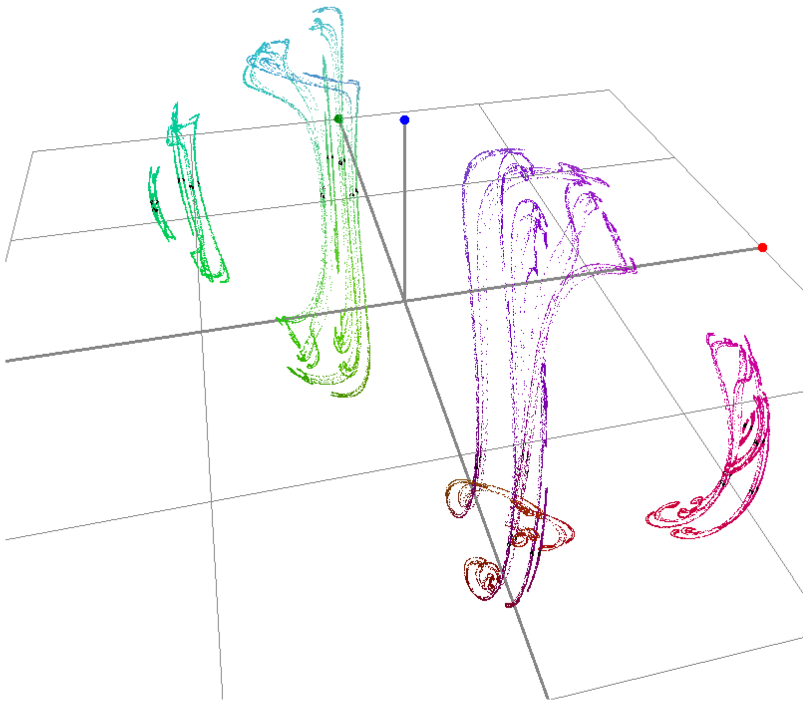


Looking down the Y axis



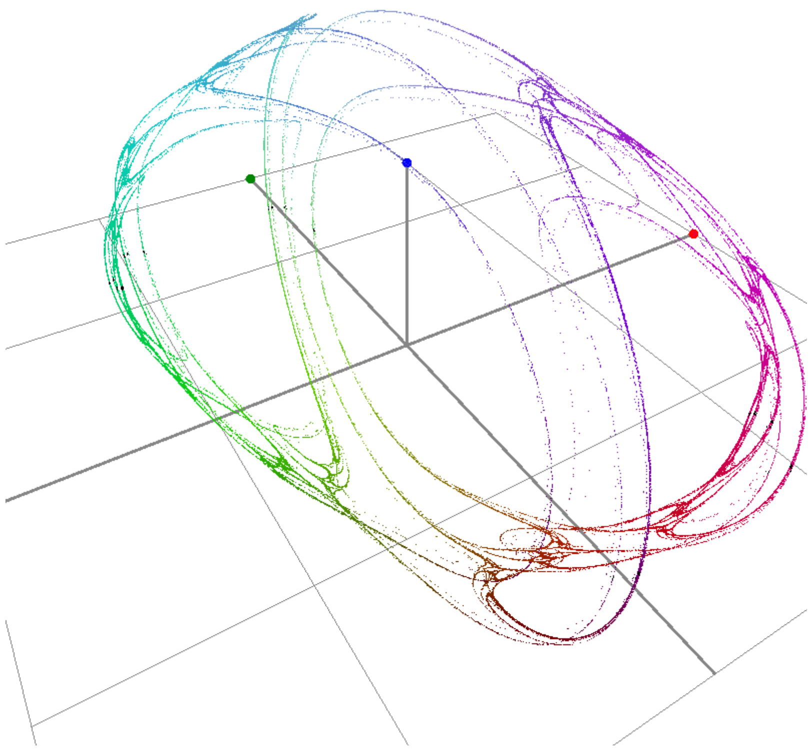
Looking down the Z axis

Another surprising thing about Linton sets is their apparent irregularity. Like Julia sets, Linton sets have antipodal symmetry. They also have a reflection symmetry which (if we restrict ourselves to values of C in which $C.z = 0$) is reflected in the XY plane. But apart from these symmetries, the detailed structure is incredibly complex. This is surprising because the fundamental rules from which the set is generated (i.e. the square root and addition rules) are about as simple as they can be, and it is the simplicity of these rules in the complex plane which causes Julia sets and the Mandelbrot set itself to have so much repetitive detail. Consider, for example the Linton set for {1, 1.6, 0}:



The curlicue scrolls at the ends of the curves are recognizable features and are reflected in the opposite curves and below the XY plane – but they are not replicated on a smaller scale elsewhere.

Here is another nice attractor – this time for {1.4, 1.4, 0}



One last comment. Three dimensional strange attractors such as the Lorentz attractor or the Tinkerbell attractor are generally very difficult to find and are often very sensitive to small changes in the parameters of the generating equations and may have quite small basins of attraction. Linton sets are, however, very robust. There is (I believe) a Linton set for every point in the universe and their basins of attraction are infinite!

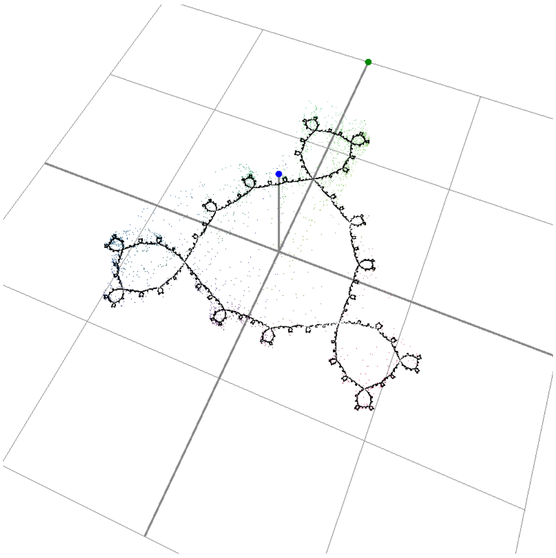
Cubic Linton Sets

So far we have only used the square root function; but it is straightforward to define the n^{th} root of a ternion in a similar way and to use the iterative equation.

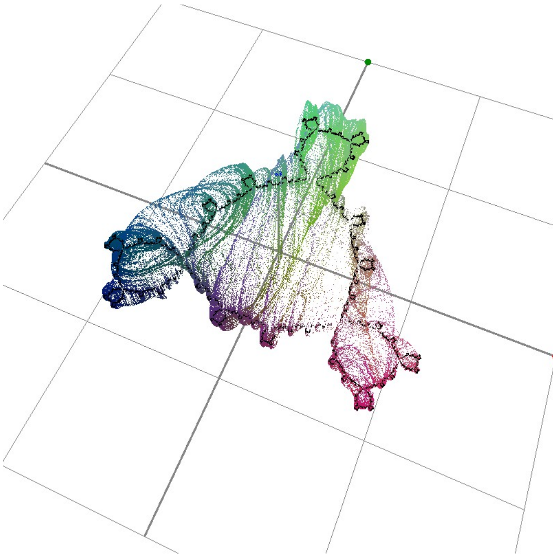
$$z' = \sqrt[n]{z - C} \tag{3}$$

When $n = 3$ there are, of course, three roots all lying in the same plane as z and the X axis and all at 120° to one another.

This is what we get when $n = 3$ and $C = \{1, 0, 0\}$. (In order to get the complete set it is necessary to apply both fuzzy arithmetic and a lot of bias towards the negative roots.)



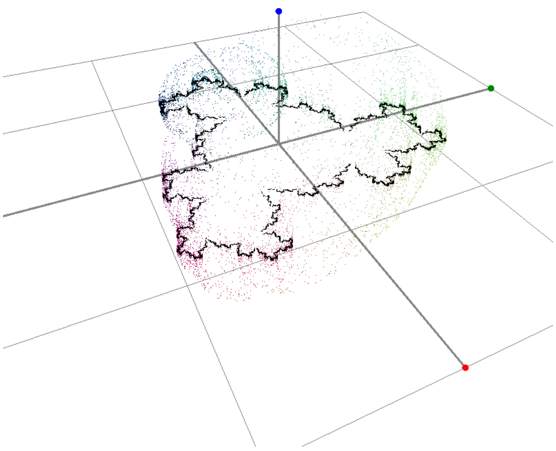
Straight



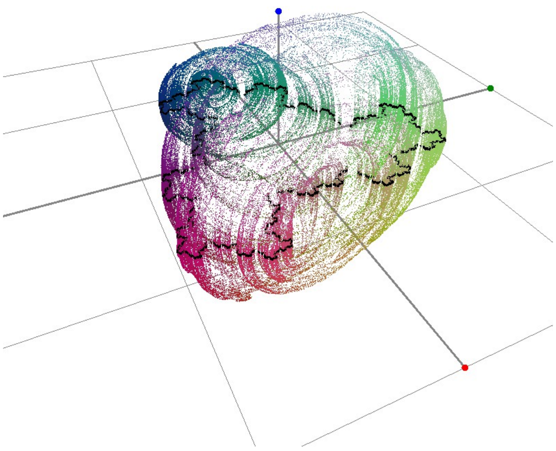
Fuzzy

As before, when we use accurate arithmetic, the Z component quickly decays and we get the standard Julia set for the point $C = (1, 0)$. Only if we apply some fuzziness do we get the full Linton set which is a sort of teddy bear with three heads!

As another example. Here is the set for the point {0.5 0.5, 0}



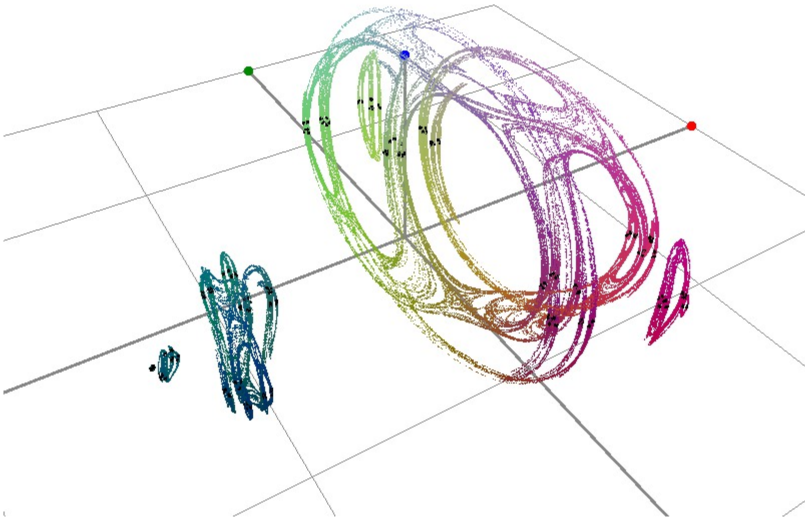
Straight



Fuzzy

This time we get a misshapen meringue!

Even at places where the standard Julia set is quite fragmentary, interesting Linton sets can be found. Here, for example, is the Linton set for {1, 0.8, 0}



The Julia set consists of three scattered groups of spots in the XY plane. In the Linton set, one of these groups (in blue) is isolated but the other two are connected by circular swirls.

Concluding remarks

How significant are these 'Linton' sets?

To be honest, I rather doubt that there will be announcements of a great new mathematical discovery in the papers and a flurry of articles on them in the mathematical journals. But insofar as they are a natural extension of the ideas of a Julia set in three dimensions, they are, I think, worth someone's attention.

It would be nice if someone with a 3D printer could print some of the solid ones for me but I have no idea how you could turn a random list of a quarter of a million three dimensional points into a format which such a printer could use!

Carr Bank: October 2022